Peak-to-Average Ratio Reduction with Tone Reservation in Multi-User and MIMO OFDM

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***Abstract*—We extend Tone Reservation for peak-to-average ratio reduction in MIMO-multiuser OFDM scenarios. First, we consider a multi-user BC (broadcast) situation where a precoding is applied at every carrier. This is considered to be a very demanding situation for peak-to-average ratio reduction. Tellado's Tone Reservation, however, is especially suited for this situation, as well. It reserves all spatial dimensions of the reserved carriers and has hence not to take into account possible implications resulting from precoding. Secondly, for a point-to-point (pt2pt) MIMO-OFDM scenario, we assume that the last eigenchannel is too weak to be used for transmission, thus, reserving it will offer redundancy for peak-to-average ratio reduction. The algorithm in both cases completely operates in time domain and is shown to additionally profit from multiple spatial dimensions when iteratively only the highest peak spatial components are processed.**

***Index Terms*—PAR reduction, Tone Reservation, Tomlinson-Harashima precoding, Broadcast, MIMO-OFDM**

I. Introduction

Multicarrier modulation is known to suffer from high peak-to-average ratio, which lead to quite some different approaches for PAR reduction. Here, we restrict ourselves to only mention Selected Mapping (SLM) [1], [2], Partial Transmit Sequences (PTS) [2], and Tone Reservation (TR) [3], [4]. SLM uses different phase rotation vectors which are applied and the sequence with the best PAR is finally transmitted. PTS subdivides the DFT domain vector into sub-blocks, applies an IFFT onto these sub-blocks padded with zeros. Phase rotations are applied onto these sub-blocks to stepwise optimize the PAR. Both solution are very complex, since many FFTs have to be applied. A trellis-shaping variant of PTS has been proposed in [5] to reduce the complexity. However, the least complexity is offered by TR originally proposed by Tellado [3], with an oversampled variant proposed by the author in [4]. Due to the extremely low complexity, TR and its oversampled variant are heavily applied in practice. In here, we only show results without oversampling, but results in [4] show that differences are small when filter responses are taken into account.

For this paper, we adopted channel assumptions of a work by Siegl and Fischer [6], who investigated especially SLM for multi-user OFDM. Since SLM applies phase shift vectors before the IFFT at the transmitter, these rotations would then conflict with the precoding matrices. Hence, only the same rotation vector can be applied to all spatial dimensions, which leads to an inefficient use of phase rotation vectors.

The channel studied in [6] can be described by a matrix polynomial $\mathbf{H}(z) = \sum_{k=0}^{l_H-1} \mathbf{H}_k z^{-k}$, where $\mathbf{H}_k$ is an $N_R \times N_T$ matrix containing complex fading coefficients from an equivalent complex baseband model between all transmit and receive antennas. Especially, the paper concentrates on the BC case with single receive antennas and U users, which sum up to $N_T = U$ antennas at the receivers. Also in here, we use N_T for all receive antennas, not the number of receive antennas per user. The length of the channel used is $l_H = 5$ (length of the cyclic prefix of OFDM is 4). The channel coefficients are i.i.d. complex Gaussian distributed with zero mean and variance $1/l_H$. The number of carriers we also chose to be $N = 128$.

In Section II, we shortly recall precoding for multi-user BC and Section III provides the TR algorithm. Section IV discusses the advantages of multi-user TR and provides simulation results. In Section V, we will then go over to the point-to-point MIMO case utilizing unused eigenchannels. Corresponding simulation results follow in Section VI. Section VII provides a short summary.

II. Precoding for Multi-User Broadcast

For Tomlinson-Harashima precoding for BC downlink, one first considers the conjugate transpose[1] of the channel matrix [7] and applies the QR decomposition, i.e.,

$$\mathbf{H}^{\mathrm{H}} = \mathbf{QR} \,. \tag{1}$$

We denote the input to the channel as $\tilde{\mathbf{x}}$, obtained from the information vector $\mathbf{x}$ by appropriate preprocessing, which is now going to be analyzed. Then the following equation holds:

$$\mathbf{y} = \mathbf{R}^{\mathrm{H}}\mathbf{Q}^{\mathrm{H}}\tilde{\mathbf{x}} + \mathbf{n} \quad . \tag{2}$$

$\mathbf{Q}$ is a unitary matrix. When defining $\mathbf{x}'$ such that $\tilde{\mathbf{x}} = \mathbf{Q}\mathbf{x}'$, Eq. (2) becomes

$$\mathbf{y} = \mathbf{R}^{\mathrm{H}}\mathbf{x}' + \mathbf{n} \,. \tag{3}$$

[1] H denotes Hermitian, i.e., conjugate transpose

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For the information vector $\mathbf{x}$, interference-free reception is achieved if the following equation holds:

$$\mathrm{diag}\left(\mathbf{R}^{\mathrm{H}}\right)\mathbf{x} = \mathbf{R}^{\mathrm{H}}\mathbf{x}' , \quad \text{i.e.,} \tag{4}$$

$$\begin{bmatrix} r_{11}x_1 \\ r_{22}x_2 \\ \vdots \\ r_{UU}x_U \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ r_{21} & r_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_{U1} & r_{U2} & \cdots & r_{UU} \end{bmatrix} \cdot \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_U' \end{bmatrix}$$

Due to the triangular structure of the matrix $\mathbf{R}^{\mathrm{H}}$, the equation leads to the following precoding operation:

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 - \frac{r_{21}}{r_{22}}x_1' \\ &\vdots \\ x_U' &= x_U - \frac{r_{U,U-1}}{r_{UU}}x_{U-1}' - \cdots \frac{r_{U,1}}{r_{UU}}x_1' \, . \end{aligned}$$

The peak power is limited by the typical modulo operation of Tomlinson-Harashima precoding leading to

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= \Gamma_{M_2}\left[x_2 - \frac{r_{21}}{r_{22}}x_1'\right] \\ &\vdots \\ x_U' &= \Gamma_{M_U}\left[x_U - \frac{r_{U,U-1}}{r_{UU}}x_{U-1}' - \cdots \frac{r_{U,1}}{r_{UU}}x_1'\right] \, . \end{aligned} \tag{5}$$

Note that $\Gamma_{M_i}[x]$ is a two-dimensional modulo operation, i.e., it can be rephrased with one-dimensional modulo operations,

$$\Gamma_{\sqrt{M_i}}[x] = \Gamma^{1D}_{\sqrt{M_i}}[\mathrm{Re}(x)] + j\Gamma^{1D}_{\sqrt{M_i}}[\mathrm{Re}(x)] \tag{6}$$

with

$$\Gamma^{1D}_{\sqrt{M_i}}[x] = x - \sqrt{M_i}d\left\lfloor \frac{x + \frac{\sqrt{M_i}d}{2}}{\sqrt{M_i}d} \right\rfloor \quad , \tag{7}$$

where $\sqrt{M_i}$ is the PAM constellation size corresponding to an M_i-QAM of user i, d is the constellation point spacing, and x is the real/imaginary value.

The simulations results presented later in this work will also hold for a pure point-to-point MIMO or a multi-user MIMO system. A point-to-point MIMO system may be described by applying a singular value decomposition

$$\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H, \tag{8}$$

which would mean a unitary preprocessing matrix $\mathbf{V}$ and post-processing $\mathbf{U}^H$. Hence, the preprocessing matrices $\mathbf{Q}$ and $\mathbf{V}$ are similar in their behavior.

III. Tone Reservation

Tone Reservation [3] reserves some carriers to generate an impulse-like function $\mathbf{p}$, which is then used in time domain to iteratively reduce peaks down to a certain target amplitude x_{target}. The steps are shown subsequently, where the central operation 5) is formulated for the complex case. $\mathbf{p} \rightarrow m$ is the cyclically shifted impulse function to the position m of the peak position. α is the step size.

Tone Reservation algorithm

1) Initialize $\mathbf{X}$ to be the DFT-domain information vector when the reserved bins are set to zero.
2) Initialize the time domain solution $\mathbf{x}^{(0)}$ to $\mathbf{x}$, obtained as the IFFT of $\mathbf{X}$
3) Find the value $x_m^{(i)}$ and location m for which $|x_m^{(i)}| = \max_k |x_k^{(i)}|$.
4) If $|x_m^{(i)}| < x_{\text{target}}$ or $i > i_{\max}$ then stop and transmit $\mathbf{x}^{(i)}$, otherwise
5) Update the time-domain vector:[2]

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \alpha \cdot (x_m^{(i)} - e^{j\mathrm{arc}(x_m^{(i)})} \cdot x_{\text{target}}) \cdot (\mathbf{p} \rightarrow m) \tag{9}$$

$i := i + 1$ and go to Step 3.

Practically, an oversampled version of the algorithm has to be applied, since filter functions always present after the algorithm will otherwise almost nullify the expected gain. Such an oversampled version is, e.g., described in [4] where a set of pairs of oversampled and non-oversampled impulsive functions are used. The number of pairs is equal to the oversampling factor and the filter functions modeled inside the TR algorithm controlling the iterations and in parallel computing also the non-oversampled time-domain function to be finally transmitted through the real filters. Since such an oversampled variant is known to almost preserve the gain of the original non-oversampled TR, in here, we show the non-oversampled results.

IV. Multi-User Tone Reservation

A. Advantages in Multi-User TR

The structure of multi-user TR is shown in Fig. 1, when the algorithm is applied separately for each antenna. However, one may check for the biggest peak at all antennas for each iteration of the algorithm. This improves the efficiency and hence the performance. The time-domain operations of the algorithm only influence the reserved carriers that do not carry data. The precoders for these carriers need actually not be determined at all, since these frequencies will also not be considered at the receiver, anyhow.

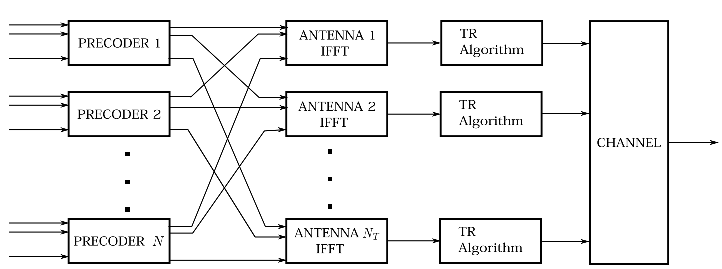

Fig. 1. Multi-user Tone Reservation

[2] arc: arcus=phase

B. Simulation Results

In the following two figures, we show simulation results for multi-user BC with 4 transmit antennas and 4 receivers with a single antenna, each. For simplicity, as average power, we refer to the one before adding reduction signals to reserved carriers. This means, as usual, we do not recompute the average power which means our "PAR" is solely a normalized measure for the peak power. The effect on the average power is indeed not significant. Like most authors, we hence show the CCDF of this kind of PAR, considering the statistics of the peak per OFDM symbol.

Independent iterations per antenna and alternatively, always processing the biggest peak among all antennas (joint processing) are shown, clearly outlining the advantage of the joint procedure. Figures 2 and 3 show the differences with the number of iterations at 5 % and 10 % reserved carriers, respectively. In the joint procedure, the total fourfold number of iterations is, of course, applied to all antennas together. The target PAR was intentionally chosen very low as 7 dB to nicely recognize the differences.

We observe gains of 4 dB and 5.7 dB at a CCDF of 10^{-6} with 5 and 10 % reserved carriers, respectively, and 12 iterations per antenna and joint processing.

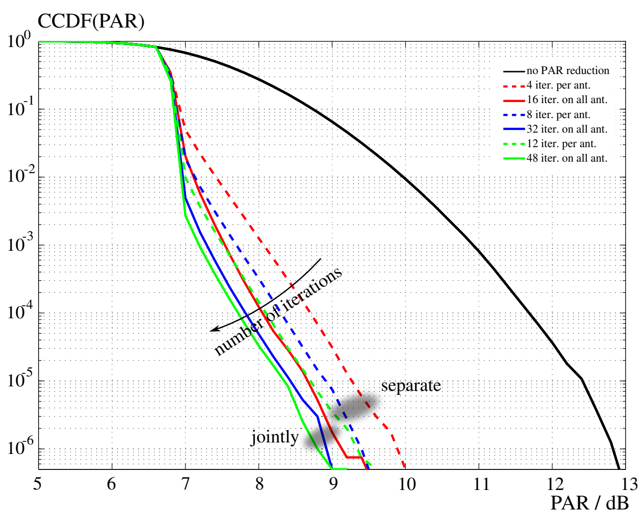

Fig. 2. CCDF of the PAR of Tone Reservation for 4×4 multi-user broadcast with 5 % redundancy depending on the number of iterations carried out separately per antenna or with joint processing

With 5 % redundancy, the reserved carrier positions were 2,7,27,30,40,98,124 using a constant real value for all of them, i.e., the spiky function was determined by only modifying the positions of reserved carriers. This spiky function corresponding to these reserved carriers is shown in Fig. 4. This is just one of many possible almost random choices.

In the following section, we will discuss MIMO approaches based on the singular value decomposition (SVD). The precoding matrix applied there is also unitary. This means every results so far is also applicable to this case. Nevertheless, there is also another option, which we will describe subsequently.

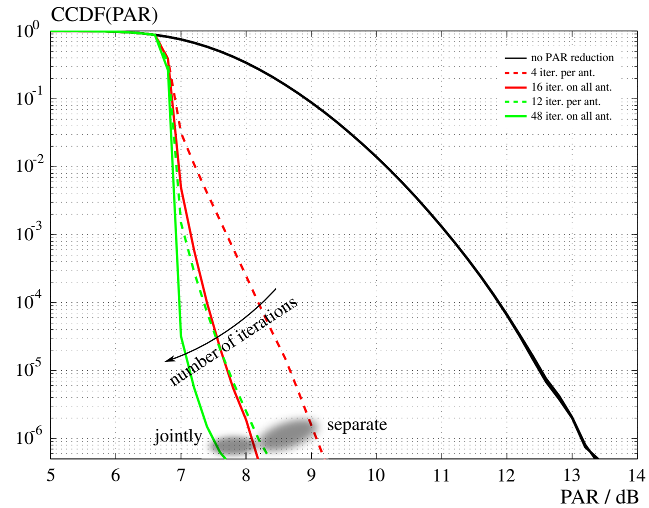

Fig. 3. CCDF of the PAR of Tone Reservation for 4×4 multi-user broadcast with 10 % redundancy depending on the number of iterations carried out separately per antenna or with joint processing

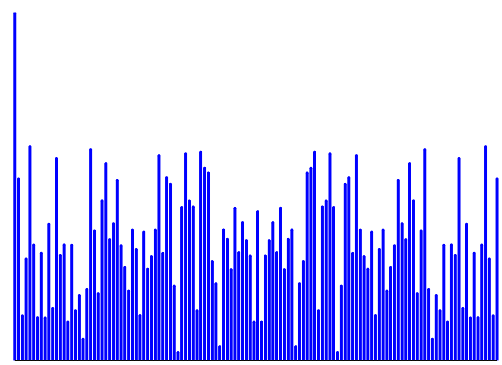

Fig. 4. Absolute values of the spiky time-domain vector used in the iterations of the Tone Reservation algorithm

V. Peak-to-Average Ratio Reduction in Point-to-Point MIMO-OFDM by Reserved Eigenchannel

A. Problem formulation

Using an SVD, the channel matrix $\mathbf{H}(n)$ in DFT domain at carrier n of a point-to-point MIMO-OFDM system is rephrased as

$$\mathbf{H}(n) = \mathbf{U}(n) \cdot \mathbf{\Lambda}(n) \cdot \mathbf{V}^H(n) \;, \tag{10}$$

where $\mathbf{U}(n)$ and $\mathbf{V}(n)$ are unitary post-processing and pre-processing matrices and $\mathbf{\Lambda}(n)$ is a diagonal matrix of the singular values of $\mathbf{H}(n)$, i.e.,

$$\mathbf{\Lambda}(n) = \begin{pmatrix} \sigma_{1,n} & 0 & 0 & 0 \\ 0 & \sigma_{2,n} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_{i,n} \end{pmatrix} . \tag{11}$$

An important property of this matrix is that, when we go down diagonally, the singular values decrease. Typically, the last singular values are so small that the corresponding eigenchannels are hardly suited for data transmission. Not using them would offer redundancy for peak-to-average ratio (PAR) reduction without a lot of cost in data rate. Throughout this paper, it is assumed that the last eigenchannel is too small and is thus reserved.

B. System Model, Channel Diagonalization, and Pre-coding

Let $\mathbf{X}(n)$ be an input vector and $\mathbf{Y}(n)$ be the output vector. At the transmitter side, we pre-multiply the signal $\mathbf{X}(n)$ by $\mathbf{V}(n)$, whereas the signal at the receiver is multiplied by $\mathbf{U}^H(n)$ to get the output $\mathbf{Y}(n)$ as shown in Fig. 5. The SVD

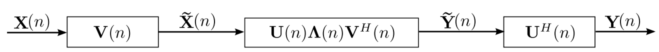

Fig. 5. SVD MIMO diagonalization

already entered in Fig. 5 leads to

$$\begin{aligned} \mathbf{Y}(n) &= \mathbf{U}^H(n) \cdot \tilde{\mathbf{Y}}(n) \\ &= \mathbf{U}^H(n) \cdot \mathbf{U}(n) \cdot \mathbf{\Lambda}(n) \cdot \mathbf{V}^H(n) \cdot \tilde{\mathbf{X}}(n) \end{aligned} \tag{12}$$

$\tilde{\mathbf{X}}$ is the product of $\mathbf{V}(n)$ and $\mathbf{X}(n)$,

$$\mathbf{Y}(n) = \mathbf{U}^H(n) \cdot \mathbf{U}(n) \cdot \mathbf{\Lambda}(n) \cdot \mathbf{V}^H(n) \cdot \mathbf{V}(n) \cdot \mathbf{X}(n) \ ,$$

where $\mathbf{U}^H(n) \cdot \mathbf{U}(n) = \mathbf{I}$ and $\mathbf{V}^H(n) \cdot \mathbf{V}(n) = \mathbf{I}$. For a 4×4 MIMO system, we write

$$\mathbf{Y}(n) = \mathbf{\Lambda}(n) \cdot \mathbf{X}(n) = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} .$$

Since the last eigenchannel has been reserved, the input data

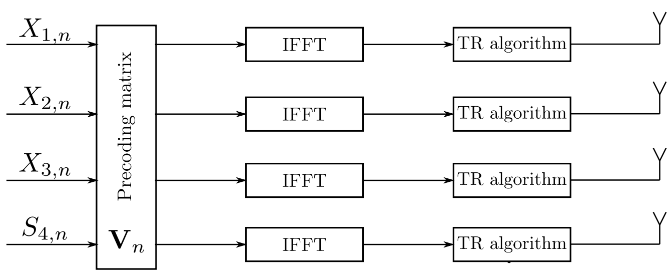

Fig. 6. Transmitter system model of MIMO-OFDM for TR algorithm

vector $\mathbf{X}(n)$ (Fig. 6) can be written as

$$\mathbf{X}(n) = \begin{pmatrix} X_{1,n} \\ X_{2,n} \\ X_{3,n} \\ 0 \end{pmatrix}$$

Now, we will generate a function, which we will again call spiky function, using the reserved eigenchannel $\mathbf{S}(n)$,

$$\mathbf{S}(n) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ S_{4,n} \end{pmatrix} .$$

The input data vector and the spiky function are pre-processed as

$$\tilde{\mathbf{X}}(\mathbf{n}) = \mathbf{V}(n) \cdot \mathbf{X}(n) = \mathbf{V}(n) \cdot \begin{pmatrix} X_{1,n} \\ X_{2,n} \\ X_{3,n} \\ 0 \end{pmatrix} \tag{13}$$

and

$$\tilde{\mathbf{S}}(n) = \mathbf{V}(n) \cdot \mathbf{S}(n) = \mathbf{V}(n) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ S_{4,n} \end{pmatrix} . \tag{14}$$

Applying the IFFT modulator, i.e., $\tilde{\mathbf{x}}^T = \mathbf{F^{-1}}\tilde{\mathbf{X}}^T$ and $\tilde{\mathbf{s}}^T = \mathbf{F^{-1}}\tilde{\mathbf{S}}^T$, where $\mathbf{F^{-1}}$ is the IFFT matrix with elements $w_{i,j} = \frac{1}{\sqrt{N}} e^{j2\pi(i-1)(j-1)/N}$ and $\tilde{\mathbf{x}}$ consist of columns $\tilde{\mathbf{x}}(n)$, $n = 0, ..., N-1$ (likewise the other matrices $\tilde{\mathbf{X}}$, $\tilde{\mathbf{s}}$, $\tilde{\mathbf{S}}$). This spiky function $\tilde{\mathbf{s}}^T$ is then iteratively added to the original function $\tilde{\mathbf{x}}^T$ in the time domain for PAR reduction. The two sum up to

$$\begin{aligned} \tilde{\mathbf{x}}^T + \tilde{\mathbf{s}}^T = \mathbf{F}^{-1}[&\mathbf{V}(0)(\mathbf{X}(0) + \mathbf{S}(0)) \ldots \\ &\mathbf{V}(N-1)(\mathbf{X}(N-1) + \mathbf{S}(N-1))]^T \ . \end{aligned} \tag{15}$$

The PAR after the algorithm is defined as

$$\text{PAR} = \frac{\max_{\forall_\mu, \forall_k} |x_{\mu,k} + s_{\mu,k}|^2}{\sigma_A^2} , \tag{16}$$

where μ is the transmit antenna, k is the sample index, and $\sigma_A^2 = E_{\forall_\mu, \forall_n}\{|x_{\mu,n}|^2\}$, may be chosen to be the average power without any PAR measures, i.e., with an unused spatial dimension. Now, our goal will be to design $\mathbf{S}$ such that it will reduce the peak-to-average power ratio to a certain target value x_{target}.

C. Designing a Spiky Function

An optimum prototype spiky function would mean a spike at time zero resulting from a constant in frequency domain at the corresponding antenna μ. Herein, we will generate four spiky functions, one at each antenna (since a 4×4 PtP system is considered). First let us assume that we like to produce a spiky function at one antenna only, not caring about the others for now.

In Eq. (14), every component (column) of the four spatial dimensions is multiplied by $\mathbf{V}(n)$. Essentially, (14) cuts out the last column of $\mathbf{V}(n)$.

A spiky function at time zero would mean a constant in frequency domain at the corresponding antenna μ, i.e., all ones for example. Now, we can easily compute the necessary $S_{4,n}$, since we know the weighting factor out of the last column of $\mathbf{V}(n)$ that corresponds to the selected antenna, i.e., one chooses

$$S_{4,n} = 1/V_{\mu,4}(n) \ . \tag{17}$$

We do not know, of course, how the other antennas are affected at the same time, however, we will select antenna μ with the highest peak. Using (16) and applying an IFFT, we obtain the modification matrix in time domain for all antennas and all times $0, 1, \ldots, n$ by

$$\mathbf{s}_\mu = \mathbf{F}^{-1} \cdot \mathbf{V} \cdot \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ S_\mu(0) & \ldots & S_\mu(n-1) \end{bmatrix} . \quad (18)$$

A spiky function corresponding to $S_\mu(n) = 1/V_{1,4}(n)$, e.g., a spike at the first antenna is as shown in Fig. 7.

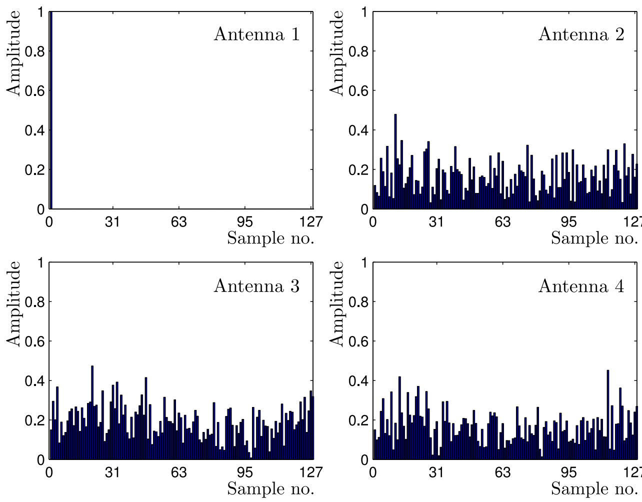

Fig. 7. Absolute values of spiky time domain vectors, with a highest peak at the 1st antenna

VI. Simulation Results

For simulation we have considered a 4×4 MIMO-OFDM system also with 128 carriers. The channel matrix described earlier is used, however, with a channel length $l_H = 25$.
From our multi-user TR results, we know that a joint search of peaks on all antennas jointly is favorable. Here, additionally, one should be aware of a possible peak increase at other antennas during the reduction at one of them, which can easily be concluded from Fig. 7. The effect on other antennas become more pronounced for very short channels due to stronger dependencies in DFT domain. Figure 8 shows simulation results, where we shifted the curves, i.e., modified the reference average power such that the unprocessed would be co-located with the one in Fig. 2 to simplify comparisons. We applied 48 iterations in total and obtain a gain of around 2.5 dB at a CCDF of 10^{-5}. The method is much more sensitive to the choice of the number of iterations, the step size, and the target value, compared to the conventional TR discussed beforehand. Easily, non-converging situations can result with a flooring of the CCDF.

VII. Summary

We have shown that Tellado's Tone-Reservation method for PAR reduction can directly be applied to multi-user broadcast and point-to-point MIMO-OFDM. Multi-user broadcast reserves some subcarriers, thus, any precoding operation present

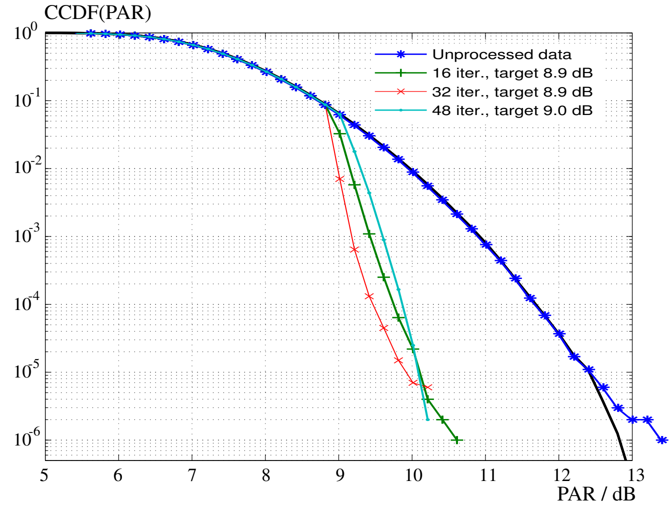

Fig. 8. CCDF of the Tone Reservation for a 4x4 Pt2Pt MIMO-OFDM depending on the number of iteration with joint search

in downlink is only of relevance for the data carriers, not the ones used for PAR reduction. Especially, when applying the reduction steps of the algorithm to the spatial channels with the currently highest peaks leads to significant performance advantages compared to blindly applying the algorithm for each antenna separately.
As an alternative for point-to-point MIMO-OFDM, we reserved the weakest eigenchannels which are then used to generate spiky function for PAR reduction. Also there, a joint search algorithm for the peaks on all antennas is proposed.
We conclude that Tone Reservation, usually in its oversampled variant, is also the method of choice in multi-antenna systems when low complexity is the selection criterion.

Acknowledgement

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