RS codes, OFDM, and MIMO

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***Abstract*— OFDM makes use of the fact that a DFT matrix diagonalizes a Toeplitz matrix, representing the convolution by a channel impulse response. Similarly, a Singular-Value Decomposition (SVD) is used to diagonalize a MIMO channel by applying unitary pre- and post-processing matrices. OFDM with a set of consecutively unused carriers can be regarded as a Reed-Solomon code over the complex number field. In the same way, SVD provides an analog code with similar properties.**

Furthermore, a combined time and DFT-domain encoding is presented which offers RS redundancy in time and DFT domain guided by the uncertainty principle.

I. Introduction

We concentrate on so-called analog codes, i.e., codes over complex numbers. However, time-frequency coding does not dependent on the actual field choice.

Analog codes in the sense of Reed-Solomon or BCH Codes over the complex number field have first been investigated by Wolf in early 1983 [1], [2]. OFDM (Orthogonal Frequency Division Multiplexing) and DMT (Discrete MultiTone) can actually be regarded as RS and BCH codes [4], respectively.

For MIMO processing, the singular-value decomposition (SVD) has been introduced as one tool for two-sided processing for the transmission between antenna arrays [3] or for bundled cables [5].

In this work, we will unveil similarities in both applying the DFT for OFDM and the SVD for MIMO, diagonalizing Toeplitz or arbitrary matrices using unitary matrices. Furthermore, we will discuss combinations between time, frequency, and spatial coding.

The following section II will rehearse the definition of Reed-Solomon codes again together with its minimum Hamming distance. This prepares for a similar argumentation in the case of SVD-based 'coding' in Section III. Both approaches are combined in Section IV to represent MIMO-OFDM transmission, i.e., space-frequency RS-like coding. Section V gives some hints regarding the decoding of RS-like space-time codes. Finally, Section VI looks into time-frequency coding with RS codes, shortly, before summarizing the paper in Section VII.

II. Reed-Solomon Codes and OFDM

RS codes are commonly defined as follows.

Definition 2.1: A **Reed-Solomon (RS) code** of length N and minimum Hamming distance d_{Hm} is a set of vectors, whose components are the values of a polynomial $C(x) = x^l \cdot C'(x)$ of degree$\{C'(x)\} \leq K-1 = N - d_{Hm}$ at positions z^k, with z being an element of order N from an arbitrary number field.

$$c = (c_0, \ldots, c_{N-1}) \ , \ \ c_i = C(x = z^i) \qquad (1)$$

w_{Hm} and d_{Hm}, the minimum Hamming weight and distance, respectively, are known to be

$$w_{Hm} = \min ||c||_0 = d_{Hm} = N-(K-1) = N-K+1 \ . \qquad (2)$$

Since the samples are chosen to be powers of an element of order N, i.e., z^k, where z would be $e^{j2\pi/N}$ in the complex case, the equivalent description is known to be

$$c_i = z^{il} \cdot \sum_{k=0}^{N-1} C_k z^{ik} \ , \quad i = 0, \ldots, N-1, \qquad (3)$$

with $C_{k+l \text{ modulo N}} = 0$ for $K \leq k \leq N-1$. For $l = 0$, (3) is a DFT, which can as well be formulated as

$$(c_0, c_1, \ldots c_{N-1}) = 1/\sqrt{N} \cdot (C_0, C_1, \ldots, C_{N-1}) \cdot \mathbf{W} \qquad (4)$$

with matrix elements $W_{ij} = z^{ij}$. We introduced the factor $1/\sqrt{N}$ to make it a unitary transform.

Usually, the minimum Hamming distance is achieved by inspecting the number of linearly independent columns of the parity-check matrix. The generator matrix, however, can be used as well. Then, the minimum Hamming weight (distance) of $N-K+1=M+1$ follows from $K\times K$ non-zero minors from arbitrary columns of the $K\times N$ generator matrix. In our case, we extract the generator matrix as (cyclically) consecutive rows of the DFT matrix $\mathbf{W}$.

Theorem 2.1: Any minor $|\mathbf{F}|$ of (any) size $K\times K$ of an $N\times N$ Fourier (DFT) matrix $\mathbf{W}$ with components $W_{k,i}=z^{ik}$, $z=e^{\pm j2\pi/N}$ and cyclically adjacent rows (or columns) is nonzero.

The non-singularity of the considered sub-matrices ensures that at most $K-1$ zeros can be achieved, leaving at least $N-K+1$ non-zero values in time domain, which is then the minimum Hamming weight (distance).

OFDM is known to consist of an IFFT and cyclic prefix (CP) extension at the transmitter and CP elimination and FFT at the receiver. The CP makes the channel convolution to appear as cyclic and the IFFT/FFT pair finally diagonalizes the channel. Practically, for band separation purposes or, in case of adaptive bit-loading, at low signal-to-noise ratios, usually, some consecutive carriers are left unused, thereby realizing an analog RS code.

III. An SVD Code

A well-known tool in wireless and wireline MIMO communication is the singular value decomposition which diagonalizes an arbitrary channel matrix $\mathbf{H}$ in the form[1]

$$\mathbf{H}=\mathbf{V}\mathbf{D}\mathbf{U}^H\ . \tag{5}$$

We assume row vectors with N_T and N_R as the number of sources and receivers (antennas in wireless), respectively. Then the dimensions would be $N_T\times N_R$, $N_T\times N_T$, $N_T\times N_R$, $N_R\times N_R$ for $\mathbf{H}$, $\mathbf{V}$, $\mathbf{D}$, $\mathbf{U}$, respectively. For simplicity, we assume $N_T=N_R=N$.

If a transmit vector $\mathbf{c}$ is preprocessed by $\mathbf{V}^H$, and the receiver provides a post-processing $\mathbf{U}$, then only the diagonal matrix $\mathbf{D}$ remains, which represents the independent spatial channels. The pre- and post-processing matrices are unitary, just like the DFT matrix used to define analog RS codes. Typically, in wireless channels, there are dominant eigenchannels and also very poor ones, that are often not used at all. Thus, we practically have a similar situation as with OFDM, i.e., analog RS-like codes, although the 'codeword' length would typically be small. Nevertheless, the random nature of the channel matrix will ensure the same Hamming distance as with an RS code.

Theorem 3.1: If the coefficients of a random channel matrix are statistically independent and chosen randomly according to some continuous distribution, the probability of a zero minor of size $K\times K$ of the preprocessing matrix $\mathbf{V}^H$ of size $N\times N$ goes to zero.

This in turn ensures the minimum Hamming distance of the resulting transmitted codeword to be $N-K+1$ just as in the case of RS codes. We can furthermore loosen the requirement to have a cyclically consecutive syndrome as necessary in RS codes. However, typically the singular values are ordered leaving the smaller ones at the end. The corresponding eigenchannels will be the ones discarded. Every $K\times K$ minor, omitting arbitrary rows and columns, will be non-zero with a probability approaching one.

IV. A Combined DFT/SVD Code

In MIMO-OFDM applications, we actually make use of OFDM for every spatial channel, meaning a channel matrix $\mathbf{H}_i$ for every ith carrier. Formally, this is described by an overall channel matrix in frequency (N carriers) and spatial ($L\times L$ MIMO) dimensions written as a blocked matrix in the form

$$\mathbf{H}_B=$$

$$\begin{pmatrix} \begin{smallmatrix} H_{11}^0 \cdot H_{1L}^0 \\ \cdot \quad \cdot \\ H_{L1}^0 \cdot H_{LL}^0 \end{smallmatrix} & & & \\ & \begin{smallmatrix} H_{11}^1 \cdot H_{1L}^1 \\ \cdot \quad \cdot \\ H_{L1}^1 \cdot H_{LL}^1 \end{smallmatrix} & & \\ & & \ddots & \\ & & & \begin{smallmatrix} H_{11}^{N-1} \cdot H_{1L}^{N-1} \\ \cdot \quad \cdot \\ H_{L1}^{N-1} \cdot H_{LL}^{N-1} \end{smallmatrix} \end{pmatrix} . \tag{6}$$

We can apply a block-wise SVD leading to

$$\mathbf{H}_B=\mathbf{V}_B\mathbf{D}_B\mathbf{U}_B^H\ , \tag{7}$$

where all matrices are block diagonal ($\mathbf{D}_B$ is, of course, strictly diagonal). A similar approach has already been described in [6]. Writing the IDFT matrix $\mathbf{W}$ for the L spatial channels also in a blocked form, we obtain the MIMO-IDFT matrix

[1] H denotes Hermitian, i.e., conjugate transpose.

$$\mathbf{W}_B = \mathbf{W} \otimes \mathbf{I} =$$

$$\begin{pmatrix}
1 & \cdot & 0 & 1 & \cdot & 0 & & & \\
\cdot & & \cdot & \cdot & & \cdot & \cdots & & \\
0 & \cdot & 1 & 0 & \cdot & 1 & & & \\
1 & \cdot & 0 & z^1 & \cdot & 0 & & & \\
\cdot & & \cdot & \cdot & & \cdot & & & \\
0 & \cdot & 1 & 0 & \cdot & z^1 & & & \\
 & & & \vdots & & & \ddots & & \\
 & & & & & & z^{(N-1)^2} & \cdot & 0 \\
 & & & & & & \cdot & & \cdot \\
 & & & & & & 0 & \cdot & z^{(N-1)^2}
\end{pmatrix} \tag{8}$$

In total, applying both the IDFT and the SVD, the overall preprocessing will be

$$\mathbf{V}_B^H \cdot (\mathbf{W} \otimes \mathbf{I}) \ . \tag{9}$$

SVD and OFDM (analog RS coding) may be seen as a serial concatenation (see Fig. 4 of [6] for a MIMO-OFDM block diagram), which would give rise to possibilities of iterative decoding in frequency and spatial domains. The overall Hamming distances would result to be the product of the individual Hamming distances assuming an equal number of unused carriers and spatial components.

One may argue that applying the DFT and SVD separately in MIMO-OFDM may not make too much sense, since the SVD itself is, of course, able to diagonalize arbitrary matrices. However, the SVD only needs to be applied per frequency to matrices of dimension $L \times L$, instead of to matrices of dimension $NL \times NL$ in the overall SVD case. Furthermore, the well-known advantages regarding equalization of multicarrier signals are, of course, also valid for MIMO-OFDM.

V. Some Short Remarks Regarding the Decoding

For SVD-based coding alone, error localization may not be necessary, since we are talking about MIMO systems of, e.g., size 4×4, which would allow to try all possible patters and decide for the best fit in a least-squares sense.

The least-squares decoding follows [4]. Let $\mathcal{L}_{ef}$ be the set of error-free received symbols at the receiver. A codeword $\mathbf{C}$ in 'transform' domain with K nonzero components needs to be recovered from a received vector $\mathbf{r}_{ef}$ containing only those components that are free of impulsive noise. We can write the squared error as

$$(\mathbf{r}_{ef} - \mathbf{C}\mathbf{A})(\mathbf{r}_{ef} - \mathbf{C}\mathbf{A})^H \ , \tag{10}$$

where $\mathbf{A}$ is a matrix representing the cascade of preprocessing by DFT and/or SVD and the channel. The squared error is minimized by the least-squares solution

$$\mathbf{C} = \mathbf{r}_{ef} \cdot \mathbf{A}^H (\mathbf{A}\mathbf{A}^H)^{-1} \ . \tag{11}$$

$\mathbf{A}$ contains the rows of the DFT/SVD matrix that correspond to the information positions and only those columns that correspond to the error-free positions in the received vector. Formally, with $\mathcal{S}$ denoting the set of syndrome positions, this means

$$\mathbf{A} = [A_{i,l}] \ , \ \ i \notin \mathcal{S} \ , \ \ l \in \mathcal{L}_{ef} \ .$$

When determining the error values from the syndrome, also there a least-squares approach may be chosen. If errors occur at the transmitter, we have

$$\mathbf{y}_s = (\mathbf{x}\mathbf{V}^H\mathbf{H}+\mathbf{e})\mathbf{U} = (\mathbf{x}\mathbf{D}\mathbf{U}^H+\mathbf{e})\mathbf{U} = \mathbf{e}\mathbf{U} \tag{12}$$

at the syndrome positions. $\mathbf{x}$ and $\mathbf{y}$ are input and output, respectively, and $\mathbf{x}$ has zero syndrome positions. $\mathbf{e}$ is the error vector at the receive antenna(s). (12) leads to the least-squares solution

$$\mathbf{e} = \mathbf{y}_s \cdot \mathbf{U}_{es}^H (\mathbf{U}_{es}\mathbf{U}_{es}^H)^{-1} \ . \tag{13}$$

$\mathbf{U}_{es}$ contains those rows and columns that correspond to the error and syndrome positions, respectively.

If the error is transmitter-sided, we likewise write

$$\mathbf{y}_s = (\mathbf{x}\mathbf{V}^H + e)\mathbf{H}\mathbf{U} = \mathbf{e}\mathbf{V}\mathbf{D} \tag{14}$$

with the least-squares solution

$$\mathbf{e} = \mathbf{y}_s \cdot (\mathbf{D}\mathbf{V})_{es}^H ((\mathbf{V}\mathbf{D})_{es} (\mathbf{D}\mathbf{V})_{es}^H)^{-1} \ , \tag{15}$$

where the index 'es' again denotes the relevant parts of the matrix corresponding to the error and syndrome positions.

VI. Time-Frequency Coding

Since the DFT is symmetric transform, one can allocate redundancy in time and DFT domain.

$$(c_0, \ldots, c_{K_T-1}, \underbrace{0, \ldots, 0}_{N-K_T}) = \tag{16}$$

$$(C_0, \ldots, C_{K-1}, \underbrace{0, \ldots, 0}_{N-K_F}) \cdot \mathbf{W} \ ,$$

where $\mathbf{W}$ is the $N \times N$ DFT matrix; K_T and K_F are the numbers of non-zero elements in time and DFT domain, respectively. The sum of zeroed positions together should be less or equal to N, leading to the uncertainty principle of the DFT, namely

$$(N - K_T) + (N - K_F) \leq N \implies K_T + K_F \geq N \tag{17}$$

The encoding to obtain time and DFT domain redundancy is no other than the usual RS encoding when using the generator polynomial $g(x)$, collecting all the linear factors corresponding to zeros in DFT domain[2]. When computing

$$c(x) = i(x) \cdot g(x) \tag{18}$$

in non-systematic encoding with an information polynomial $i(x)$, one has just to reduce the degree of this polynomial by K_T, which is also the same as shortening an RS code in information. However, when actually transmitting the nulled components, they serve as time-domain redundancy.

The mixed time-frequency coding allows for correcting periodic and distributed errors (in time domain), governed by $\lfloor (N - K_T)/2 \rfloor$ and $\lfloor (N - K_F)/2 \rfloor$, respectively. For erasure decoding, of course, this would be $N - K_T$ and $N - K_F$, respectively.

VII. Conclusions

We outlined options for space-frequency and time-frequency coding based on encoding with the DFT (Discrete Fourier Transform; Reed-Solomon code) or the SVD (Singular-Value Decomposition). The DFT is the essential component in OFDM (Orthogonal Frequency Division Multiplexing) and the SVD is one typical option for MIMO (Multiple-Input Multiple-Output) systems. We realized that the SVD has comparable properties for random MIMO channels as we know them from RS codes. The following formulas summarize the correspondence again:

$$\text{OFDM:} \quad \mathbf{W}^H \cdot \overbrace{\mathbf{W}\mathbf{D}\mathbf{W}^H}^{\text{channel}} \cdot \mathbf{W} \,, \tag{19}$$

$$\text{SVD:} \quad \mathbf{V}^H \cdot \underbrace{\mathbf{V}\,\mathbf{D}\,\mathbf{U}^H}_{\text{channel}} \cdot \mathbf{U} \,. \tag{20}$$

All matrices $\mathbf{W}$ (DFT), $\mathbf{V}$, and $\mathbf{U}$ (SVD) are unitary.

We looked into possible combinations of SVD- and DFT-based coding thereby creating a concatenated space-frequency code. Furthermore, we realized simple RS-based time-frequency encoding, by just shortening the information.

The OFDM- and SVD-based encoding is often realized in practice by omitting carriers or eigenchannels without being aware of the generated code properties.

Different decoding options, especially, as far as space-frequency coding is concerned, will be studied soon.

Almost throughout the paper, we assumed the field of complex numbers. RS codes can be defined over any field and thus, generalizations to other fields are obvious. A counterpart of the SVD for finite fields is for further study. One may think of the diagonalization by the Smith Normal Form (Invariant Factor theorem) as a possible candidate. Unfortunately, it leads to very sparse pre- and post-processing matrices.

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[2] Due to the symmetry of the underlying DFT, one could, of course, also define the generator polynomial in DFT domain to create zeros in time domain.